ON DEFORMATIONS OF FANO THREEFOLDS WITH TERMINAL SINGULARITIES

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Abstract. We study the deformation theory of a $\mathbb{Q}$-Fano 3-fold with only terminal singularities. First, we show that the Kuranishi space of a $\mathbb{Q}$-Fano 3-fold is smooth. Second, we show that every $\mathbb{Q}$-Fano 3-fold with only "ordinary" terminal singularities is $\mathbb{Q}$-smoothable, that is, it can be deformed to a $\mathbb{Q}$-Fano 3-fold with only quotient singularities. Finally, we prove $\mathbb{Q}$-smoothability of a $\mathbb{Q}$-Fano 3-fold assuming the existence of a Du Val anticanonical element. As an application, we get the genus bound for primary $\mathbb{Q}$-Fano 3-folds with Du Val anticanonical elements.

Contents

1. Introduction

All algebraic varieties in this paper are defined over $\mathbb{C}$.

1.1. Background and our results.

Definition 1.1. Let X be a normal projective variety. We say that X is *a $\mathbb{Q}$-Fano 3-fold* if $\dim X = 3$, X has only terminal singularities and $-K_X$ is an ample $\mathbb{Q}$-Cartier divisor.

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$\mathbb{Q}$-Fano 3-folds are important objects in the classification of algebraic varieties. Toward the classification of $\mathbb{Q}$-Fano 3-folds, it is fundamental to study their deformations.

Definition 1.2. Let X be an algebraic variety and Δ^1 an open unit disc of dimension 1. A $\mathbb{Q}$-*smoothing* of X is a flat morphism of complex analytic spaces $f\colon \mathcal{X} \to \Delta^1$ such that $f^{-1}(0) \simeq X$ and $f^{-1}(t)$ has only quotient singularities of codimension at least 3.

If X is proper, we assume that f is a proper morphism.

Remark 1.3. Schlessinger [26] proved that an isolated quotient singularity of dimension ≥ 3 is infinitesimally rigid under small deformations.

Reid ([24], [25]) and Mori [15] showed that a 3-dimensional terminal singularity can be written as a quotient of an isolated cDV hypersurface singularity by a finite cyclic group action and it admits a $\mathbb{Q}$-smoothing.

In general, a local deformation may not lift to a global deformation. However, Altınok–Brown–Reid conjectured the following in [2] 4.8.3.

Conjecture 1.4. *Let X be a $\mathbb{Q}$-Fano 3-fold. Then X has a $\mathbb{Q}$-smoothing.*

The following theorem is an answer to their question in the "ordinary" case.

Theorem 1.5. *Let X be a $\mathbb{Q}$-Fano 3-fold with only ordinary terminal singularities (See Remark 1.6). Then X has a $\mathbb{Q}$-smoothing.*

We prove a more general statement in Theorem 3.2 that implies Theorem 1.5.

Remark 1.6. A 3-dimensional terminal singularity is called *ordinary* if the defining equation of its index 1 cover is $\mathbb{Z}_r$-invariant, where $\mathbb{Z}_r$ is the Galois group of the cover. In the list of 3-dimensional terminal singularities, there are 5 families of ordinary singularities and 1 *exceptional* family of Gorenstein index 4 (See [15] Theorem 12 (2) or [25] (6.1) Figure (2)).

Previously, Namikawa [18] proved that a Fano 3-fold with only terminal Gorenstein singularities admits a smoothing, that is, it can be deformed to a smooth Fano 3-fold. Minagawa [13] proved $\mathbb{Q}$-smoothability of a $\mathbb{Q}$-Fano 3-fold of Fano index one, that is, it has a global index one cover. Takagi also treated some cases in [30] Theorem 2.1. Note that the singularities on a $\mathbb{Q}$-Fano 3-fold in their cases are ordinary.

In order to prove the $\mathbb{Q}$-smoothablity, we need the following theorem on the unobstructedness of deformations of a $\mathbb{Q}$-Fano 3-fold.

Theorem 1.7. *Let X be a $\mathbb{Q}$-Fano 3-fold. Then the deformations of X are unobstructed.*

Namikawa [18] proved the unobstructedness in the Gorenstein case and Minagawa [13] proved it for a $\mathbb{Q}$-Fano 3-fold of Fano index one. We show it for any $\mathbb{Q}$-Fano 3-fold. This theorem reduces the problem of finding good deformations to that of 1st order infinitesimal deformations.

Another fundamental problem in the classification of $\mathbb{Q}$-Fano 3-folds is to find anticanonical elements with only mild singularities. An anticanonical element is called an *elephant*. A Gorenstein Fano 3-fold with only canonical singularities has an elephant with only Du Val singularities([28], [23]). By using this fact, Mukai classified "indecomposable" Gorenstein Fano 3-folds with canonical singularities in [16]. Hence the existence of a Du Val elephant is useful in the classification. However a $\mathbb{Q}$-Fano 3-fold may not have such an good element in general. There exist examples of $\mathbb{Q}$-Fano 3-folds with empty anticanonical linear systems

or with only non Du Val elephants as in [2] 4.8.3. Nevertheless, Altınok–Brown–Reid [2] conjectured the following.

Conjecture 1.8. *Let X be a $\mathbb{Q}$-Fano 3-fold. Assume that $|-K_X|$ contains an element D.*

(1) *Then there exists a deformation $f\colon \mathcal{X} \to \Delta^1$ of X such that $|-K_{\mathcal{X}_t}|$ contains an element D_t with only Du Val singularities for general $t \in \Delta^1$.*
(2) *Moreover, a divisor $D_t \subset \mathcal{X}_t$ is locally isomorphic to $\frac{1}{r}(a, r-a) \subset \frac{1}{r}(1, a, r-a)$, where both sides are corresponding cyclic quotient singularities for some coprime integers r and a around each Du Val singularities of D_t.*

We call a deformation as above a *simultaneous $\mathbb{Q}$-smoothing* of a pair (X, D). If we first assume the existence of a Du Val elephant, we get the following result.

Theorem 1.9. *Let X be a $\mathbb{Q}$-Fano 3-fold. Assume that $|-K_X|$ contains an element D with only Du Val singularities.*

Then X has a simultaneous $\mathbb{Q}$-smoothing. In particular, X has a $\mathbb{Q}$-smoothing.

Note that we do not need the assumption of ordinary singularities as in Theorem 1.5. The motivation of Conjecture 1.8 is to treat a $\mathbb{Q}$-Fano 3-fold with only non Du Val elephants. We investigate this case in elsewhere.

A $\mathbb{Q}$-Fano 3-fold is called *primary* if its canonical divisor generates the class group mod torsion elements. Takagi [29] studied primary $\mathbb{Q}$-Fano 3-folds with only terminal quotient singularities and established the genus bound for those with Du Val elephants. Hence Theorems 1.5 and 1.9 are useful for the classification. Actually, as an application of Theorem 1.9, we can reprove his bound as follows.

Corollary 1.10. *Let X be a primary $\mathbb{Q}$-Fano 3-fold. Assume that X is non-Gorenstein and $|-K_X|$ contains an element with only Du Val singularities.*

Then $h^0(X, -K_X) \leq 10$.

Takagi expected the existence of a Du Val elephant for X such that $h^0(X, -K_X)$ is appropriately big ([29, p.37]). If we assume the expectation, Corollary 1.10 implies the genus bound as above for every primary $\mathbb{Q}$-Fano 3-fold.

1.2. **Outline of the proofs.** We sketch the proof of the above theorems on a $\mathbb{Q}$-Fano 3-fold X.

First, we explain how to prove the unobstructedness briefly. If X is Gorenstein, we have

$$\operatorname{Ext}^2(\Omega^1_X, \mathcal{O}_X) \simeq \operatorname{Ext}^2(\Omega^1_X \otimes \omega_X, \omega_X) \simeq H^1(X, \Omega^1_X \otimes \omega_X)^*$$

since ω_X is invertible and the unobstructedness is reduced to the Kodaira-Nakano type vanishing of the cohomology. However, if X is non-Gorenstein, that is, ω_X is not invertible, we can not reduce the vanishing of the Ext group to the vanishing of cohomology groups a priori and we do not have a direct method to prove the vanishing of the Ext group. Moreover, since we do not have a branched cover of a $\mathbb{Q}$-Fano 3-fold which is Fano or Calabi-Yau in the general case, we can not reduce the unobstructedness to that of such cover. We solve this difficulty by considering the obstruction classes rather than the ambient obstruction space Ext^2 and considering the smooth part. The important point is that deformations of X are bijective to deformations of the smooth part as in [11] 12.1.8 or [10] Theorem 12. The description of the obstruction by a 2-term extension as in Proposition 2.3 is a crucial tool.

In order to find a good deformation of first order, we follow the line of the proof in the case of Fano index 1 by Minagawa [13] which used [19] Theorem 1 of Namikawa-Steenbrink on the non-vanishing of the homomorphism between cohomology groups. We need a generalisation of this theorem to the non-Gorenstein setting which is Proposition 3.1. We can generalise this lemma provided that the singularity is ordinary. The generalisation of this lemma for general terminal singularities implies Conjecture 1.4.

Now, in order to find a good deformation of first order under the assumption of a Du Val elephant, we use the deformation theory of the pair of X and D where $D \in |-K_X|$. The smoothness of the Kuranishi space of X implies that the smoothness of the Kuranishi space of the pair (X, D) for $D \in |-K_X|$ (Theorem 2.10). Here we adapt the diagram of [19] Theorem 1.3 to the case (X, D). Instead of the Namikawa-Steenbrink's proposition [19] Theorem 1.1 on non-vanishing of a certain cohomology map, we use the coboundary map of the local cohomology sequence for the pair. To use such a map, we arrange a resolution of singularities of the pair which has non-positive discrepancies as in Proposition 4.1. Moreover we refine the Lefschetz theorem for class groups by Ravindra-Srinivas [21] for our cases and this Lefschetz statement plays an important role for lifting.

2. Unobstructedness of deformations of a $\mathbb{Q}$-Fano 3-fold

2.1. **Preliminaries on infinitesimal deformations.** Let X be an algebraic scheme over an algebraically closed field k. Let $\mathcal{A}$ be the category of local Artinian k-algebras with residue field k. For $A \in \mathcal{A}$, set

$$\mathrm{Def}_X(A) := \{f \colon \mathcal{X} \to \operatorname{Spec} A \mid f \text{ is a deformation of } X \text{ over } A\}/\text{isomorphism}.$$

Then it defines a functor

$$\mathrm{Def}_X \colon \mathcal{A} \to (\mathrm{Sets}).$$

Definition 2.1. We say that deformations of X are *unobstructed* if, for all $A, A' \in \mathcal{A}$ with an exact sequence

$$0 \to J \to A' \to A \to 0$$

such that $\mathfrak{m}_{A'} \cdot J = 0$, the natural restriction map of deformations

$$\mathrm{Def}_X(A') \to \mathrm{Def}_X(A)$$

is surjective, that is, Def_X is a smooth functor.

Proposition 2.2. *Let X be an algebraic scheme with a versal formal couple $(R, \hat{u})$ in the sense of Definition* 2.2.6 *in* [27]. *Set $A_m := k[t]/(t^{m+1})$ for all integers $m \geq 0$. Assume that*

$$\mathrm{Def}_X(A_{n+1}) \to \mathrm{Def}_X(A_n)$$

are surjective for all non-negative integers $n \geq 0$.

Then deformations of X are unobstructed.

Proof. For $A \in \mathcal{A}$, let $h_R(A)$ be the set of local k-algebra homomorphisms from R to A. This rule defines a functor

$$h_R \colon \mathcal{A} \to (\mathrm{Sets}).$$

Since $(R, \hat{u})$ is versal, we have a smooth morphism of functors

$$\phi_{\hat{u}} \colon h_R \to \mathrm{Def}_X$$

defined by $\hat{u}$.

Then we can see that

$$h_R(A_{n+1}) \to h_R(A_n)$$

are surjective for all n by the assumption and the versality.

By Lemma 5.6 in [4] and the assumption, we can see that h_R is a smooth functor. This implies that Def_X is smooth. □

We need the following description of the obstruction space for deformations.

Proposition 2.3. *Let k be an algebraically closed field of characteristic 0. Let X be a reduced scheme of finite type over k. Let $U \subset X$ be an open subset with only l.c.i. singularities and $\iota\colon U \to X$ an inclusion map. Assume that $\operatorname{depth} \mathcal{O}_{X,p} \geq 3$ for all scheme theoretic point $p \in X \setminus U$. Let Ω^1_U be the Kähler differential sheaf on U. Set $A_n := k[t]/(t^{n+1})$ and let*

$$\xi_n := (f_n\colon \mathcal{X}_n \to \operatorname{Spec} A_n)$$

be a deformation of X.

Then the obstruction to lift $\mathcal{X}_n$ over A_{n+1} lies in $\operatorname{Ext}^2(\Omega^1_U, \mathcal{O}_U)$.

Proof. We need to define an element

$$o_{\xi_n} \in \operatorname{Ext}^2(\Omega^1_U, \mathcal{O}_U)$$

which has a property that $o_{\xi_n} = 0$ if and only if there is a deformation $\xi_{n+1} = (f_{n+1}\colon \mathcal{X}_{n+1} \to \operatorname{Spec} A_{n+1})$ which sits in the following cartesian diagram;

$$\begin{array}{ccc} \mathcal{X}_{n+1} & \longleftarrow & \mathcal{X}_n \\ \downarrow & & \downarrow \\ \operatorname{Spec} A_{n+1} & \longleftarrow & \operatorname{Spec} A_n. \end{array} \tag{1}$$

Since the characteristic of k is zero, we have

$$\Omega^1_{A_n/k} \simeq A_{n-1}$$

as A_n-modules and an exact sequence

$$0 \to (t^{n+1}) \xrightarrow{d} \Omega^1_{A_{n+1}/k} \otimes_{A_{n+1}} A_n \to \Omega^1_{A_n/k} \to 0.$$

Let $f_{\mathcal{U}_n}\colon \mathcal{U}_n \to \operatorname{Spec} A_n$ be the flat deformation of U induced by f_n. By pulling back the above sequence by the flat morphism $f_{\mathcal{U}_n}$, we get the following exact sequence;

$$0 \to \mathcal{O}_U \to f^*_{\mathcal{U}_n}(\Omega^1_{\operatorname{Spec} A_{n+1}/k}|_{\operatorname{Spec} A_n}) \to f^*_{\mathcal{U}_n}\Omega^1_{\operatorname{Spec} A_n/k} \to 0. \tag{2}$$

Then, there is the relative cotagent sequence of a relative l.c.i. morphism $f_{\mathcal{U}_n}$ (cf. [27] Theorem D.2.8);

$$0 \to f^*_{\mathcal{U}_n}\Omega^1_{\operatorname{Spec} A_n/k} \to \Omega^1_{\mathcal{U}_n/k} \to \Omega^1_{\mathcal{U}_n/\operatorname{Spec} A_n} \to 0. \tag{3}$$

By combining the sequences (2), (3), we get the following exact sequence;

$$0 \to \mathcal{O}_U \to f^*_{\mathcal{U}_n}(\Omega^1_{\operatorname{Spec} A_{n+1}/k}|_{\operatorname{Spec} A_n}) \to \Omega^1_{\mathcal{U}_n/k} \to \Omega^1_{\mathcal{U}_n/\operatorname{Spec} A_n} \to 0. \tag{4}$$

Let

$$o_{\xi_n} \in \operatorname{Ext}^2(\Omega^1_U, \mathcal{O}_U) \simeq \operatorname{Ext}^2(\Omega^1_{\mathcal{U}_n/\operatorname{Spec} A_n}, \mathcal{O}_U)$$

be the element corresponding to the exact sequence (4).

We check that this o_{ξ_n} is the obstruction to the existence of lifting of ξ_n over A_{n+1}.

Suppose that we have a lifting $\xi_{n+1} = (f_{n+1} \colon \mathcal{X}_{n+1} \to \operatorname{Spec} A_{n+1})$ with the diagram (1). Then we can see that $o_{\xi_n} = 0$ as in Proposition 2.4.8 in [27].

Conversely, suppose that $o_{\xi_n} = 0$. Consider the following exact sequence

$$\operatorname{Ext}^1(\Omega^1_{\mathcal{U}_n/k}, \mathcal{O}_U) \xrightarrow{\epsilon} \operatorname{Ext}^1(f^*_{\mathcal{U}_n}\Omega^1_{\operatorname{Spec} A_n/k}, \mathcal{O}_U) \xrightarrow{\delta} \operatorname{Ext}^2(\Omega^1_{\mathcal{U}_n/\operatorname{Spec} A_n}, \mathcal{O}_U)$$

which is induced by the exact sequence (3). Consider

$$\gamma \in \operatorname{Ext}^1(f^*_{\mathcal{U}_n}\Omega^1_{\operatorname{Spec} A_n/k}, \mathcal{O}_U)$$

which corresponds to the exact sequence (2). It is easy to see that $\delta(\gamma) = o_{\xi_n}$. Hence there exists $\gamma' \in \operatorname{Ext}^1(\Omega^1_{\mathcal{U}_n/k}, \mathcal{O}_U)$ such that $\epsilon(\gamma') = \gamma$. The class γ' corresponds to the following short exact sequence

$$0 \to \mathcal{O}_U \to \mathcal{E} \to \Omega^1_{\mathcal{U}_n/k} \to 0$$

for some coherent sheaf $\mathcal{E}$ on $\mathcal{U}_n$. We can construct a sheaf of A_n-algebras $\mathcal{O}_{\mathcal{U}_{n+1}}$ by $\mathcal{O}_{\mathcal{U}_{n+1}} := \mathcal{E} \times_{\Omega^1_{\mathcal{U}_n/k}} \mathcal{O}_{\mathcal{U}_n}$ as in Theorem 1.1.10 in [27] with an exact sequence of sheaves of A_n-algebras

$$0 \to \mathcal{O}_U \to \mathcal{O}_{\mathcal{U}_{n+1}} \to \mathcal{O}_{\mathcal{U}_n} \to 0. \tag{5}$$

Set $\mathcal{O}_{\mathcal{X}_{n+1}} := \iota_* \mathcal{O}_{\mathcal{U}_{n+1}}$.

We need the following claim.

Claim 2.4. (i) $R^1\iota_*\mathcal{O}_U = 0$.

(ii) Let M be a finite A_n-module. Then

$$R^1\iota_*(f^*_{\mathcal{U}_n}\widetilde{M}) = 0,$$

where $\widetilde{M}$ is a coherent sheaf on $\operatorname{Spec} A_n$ associated to M.

Proof of Claim. (i) Let $p \in X \setminus U$ be a point and U_p a small affine neighbourhood of p. Put $Z_p := U_p \cap (X \setminus U)$. It is enough to show that $H^1(U_p \setminus Z_p, \mathcal{O}_{U_p \setminus Z_p}) = 0$. We have $H^2_{Z_p}(U_p, \mathcal{O}_{U_p}) = 0$ since $\operatorname{depth}_q \mathcal{O}_{X,q} \geq 3$ for all scheme-theoretic point $q \in Z_p$ by the hypothesis. Since $H^i(U_p, \mathcal{O}_{U_p}) = 0$ for $i = 1, 2$, we have $H^1(U_p \setminus Z_p, \mathcal{O}_{U_p}) \simeq H^2_{Z_p}(U_p, \mathcal{O}_{U_p}) = 0$.

(ii) We proceed by induction on $\dim_k M$.

If $M \simeq k$, then this is the first claim.

Now assume that there is an exact sequence

$$0 \to k \to M \to M' \to 0$$

of A_n-modules and the claim holds for M'. Then we have an exact sequence

$$R^1\iota_*(f^*_{\mathcal{U}_n}\widetilde{k}) \to R^1\iota_*(f^*_{\mathcal{U}_n}\widetilde{M}) \to R^1\iota_*(f^*_{\mathcal{U}_n}\widetilde{M'})$$

and the left and right hand sides are zero by the induction hypothesis. Hence $R^1\iota_*(f^*_{\mathcal{U}_n}\widetilde{M}) = 0$. □

Note that $\iota_*\mathcal{O}_U \simeq \mathcal{O}_X, \iota_*\mathcal{O}_{\mathcal{U}_n} \simeq \mathcal{O}_{\mathcal{X}_n}$ by Claim 2.4. By taking ι_* of (5), we have an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_{\mathcal{X}_{n+1}} \to \mathcal{O}_{\mathcal{X}_n} \to 0 \tag{6}$$

since $R^1\iota_*\mathcal{O}_U = 0$.

We can regard $\mathcal{O}_{\mathcal{X}_{n+1}}$ as a sheaf of A_{n+1}-algebras by the homomorphism $A_{n+1} \to A_n$. We can see that $\mathcal{O}_{\mathcal{X}_{n+1}}$ is a sheaf of flat A_{n+1}-algebras by the local criterion of flatness (cf. [7]

Proposition 2.2) and the exact sequence (6). Let $\mathcal{X}_{n+1} := (X, \mathcal{O}_{\mathcal{X}_{n+1}})$ be the scheme defined by the sheaf $\mathcal{O}_{\mathcal{X}_{n+1}}$. Then the morphism $\mathcal{X}_{n+1} \to \operatorname{Spec} A_{n+1}$ is flat and

$$\xi_{n+1} := (\mathcal{X}_{n+1} \to \operatorname{Spec} A_{n+1})$$

is a lifting of ξ_n. □

Remark 2.5. The author does not know whether the above construction of obstruction classes works for general A, A' as in Definition 2.1. However Proposition 2.2 reduces the study of unobstructedness to the case $A = A_n$, $A' = A_{n+1}$.

2.2. **Proof of the theorem.** We need the following Lefschetz type theorem.

Theorem 2.6. ([5]) *Let $X \subset \mathbb{P}^N$ be a projective variety of dimension n and $L \subset \mathbb{P}^N$ a linear subspace of codimension $d \leq n$. Assume that $X \setminus (X \cap L)$ has only l.c.i. singularities. Then the relative homotopy group satisfies*

$$\pi_i(X, X \cap L) = 0 \qquad (i \leq n - d).$$

In particular, the restriction map $H^i(X, \mathbb{C}) \to H^i(X \cap L, \mathbb{C})$ is injective for $i \leq n - d$.

Proof. It is [5] Chapter 3.1. Theorem. □

By using the obstruction class in Proposition 2.3, we can show the following theorem.

Theorem 2.7. *Let X be a $\mathbb{Q}$-Fano 3-fold. Then deformations of X are unobstructed.*

Proof. Let U be the smooth part of X. Note that $\operatorname{codim}_X X \setminus U \geq 3$ and X is Cohen-Macaulay since X has only terminal singularities. Hence X and U satisfy the assumption of Proposition 2.3. Set $k := \mathbb{C}$.

Let $\xi_n \in \operatorname{Def}_X(A_n)$ be a deformation of X

$$f_n \colon \mathcal{X}_n \to \operatorname{Spec} A_n$$

and $o_{\xi_n} \in \operatorname{Ext}^2(\Omega^1_U, \mathcal{O}_U)$ the obstruction class defined in the proof of Proposition 2.3. We show that $o_{\xi_n} = 0$ in the following.

Let ω_X be the dualizing sheaf on X. By taking the tensor product of the sequence (4) with the relative dualizing sheaf $\omega_{\mathcal{U}_n/\operatorname{Spec} A_n}$ of $f_{\mathcal{U}_n}$, we have an exact sequence

$$\begin{aligned}(7) \quad 0 \to \omega_U \to f^*_{\mathcal{U}_n}(\Omega^1_{\operatorname{Spec} A_{n+1}/k}|_{\operatorname{Spec} A_n}) \otimes \omega_{\mathcal{U}_n/\operatorname{Spec} A_n} \\ \to \Omega^1_{\mathcal{U}_n/k} \otimes \omega_{\mathcal{U}_n/\operatorname{Spec} A_n} \to \Omega^1_{\mathcal{U}_n/\operatorname{Spec} A_n} \otimes \omega_{\mathcal{U}_n/\operatorname{Spec} A_n} \to 0.\end{aligned}$$

By taking ι_* of the above sequence, we get a sequence

$$\begin{aligned}(8) \quad 0 \to \omega_X \to \iota_*(f^*_{\mathcal{U}_n}\Omega^1_{\operatorname{Spec} A_{n+1}/k}|_{\operatorname{Spec} A_n} \otimes \omega_{\mathcal{U}_n/\operatorname{Spec} A_n}) \\ \to \iota_*(\Omega^1_{\mathcal{U}_n/k} \otimes \omega_{\mathcal{U}_n/\operatorname{Spec} A_n}) \to \iota_*(\Omega^1_{\mathcal{U}_n/\operatorname{Spec} A_n} \otimes \omega_{\mathcal{U}_n/\operatorname{Spec} A_n}) \to 0.\end{aligned}$$

This sequence is exact by the following claim.

Claim 2.8. (i) $R^1\iota_*\omega_U = 0$.
(ii) $R^1\iota_*(f^*_{\mathcal{U}_n}\Omega^1_{\operatorname{Spec} A_n/k} \otimes \omega_{\mathcal{U}_n/\operatorname{Spec} A_n}) = 0$.

Proof of Claim. (i) Let $p \in X \setminus U$ be a singular point and U_p a small affine neighbourhood at p. It is enough to show that $H^2_p(U_p, \omega_{U_p}) = 0$. Let $\pi_p \colon V_p \to U_p$ be the index 1 cover of

U_p. Then we have $(\pi_p)_*\mathcal{O}_{V_p} \simeq \oplus_{i=0}^{r-1}\mathcal{O}_{U_p}(iK_{U_p})$ where r is the index of the singularity $p \in X$. Hence

$$H^2_q(V_p, \mathcal{O}_{V_p}) \simeq \bigoplus_{i=0}^{r-1} H^2_p(U_p, \mathcal{O}_{U_p}(iK_{U_p})),$$

where $q := \pi^{-1}(p)$. L.H.S. is zero by the same argument as in Claim 2.4 since $\operatorname{depth}_q \mathcal{O}_{V_p,q} = 3$. Hence we proved the first claim.
(ii) Let $f_{(n,p)} \colon \mathcal{U}_{(n,p)} \to \operatorname{Spec} A_n$ be the deformation of U_p induced from f_n. It is enough to show that

$$H^2_p(\mathcal{U}_{(n,p)}, f^*_{(n,p)}\Omega^1_{\operatorname{Spec} A_n/k} \otimes \omega_{\mathcal{U}_{(n,p)}/A_n}) = 0.$$

Set $\omega^{[i]}_{\mathcal{U}_{(n,p)}/A_n} := \iota_*\omega^{\otimes i}_{\mathcal{U'}_{(n,p)}/A_n}$, where $\mathcal{U}'_{(n,p)} := \mathcal{U}_{(n,p)} \setminus \{p\}$. We can take an index 1 cover $\phi_{(n,p)} \colon \mathcal{V}_{(n,p)} \to \mathcal{U}_{(n,p)}$ which is determined by an isomorphism $\omega^{[r_p]}_{\mathcal{U}_{(n,p)}/A_n} \simeq \mathcal{O}_{\mathcal{U}_{(n,p)}}$, where r_p is the Gorenstein index of U_p. Set $g_{(n,p)} := f_{(n,p)} \circ \phi_{(n,p)}$. Note that

$$(\phi_{(n,p)})_*(g^*_{(n,p)}\Omega^1_{\operatorname{Spec} A_n/k}) \simeq \bigoplus_{i=0}^{r-1} f^*_{(n,p)}\Omega^1_{\operatorname{Spec} A_n/k} \otimes \omega^{[i]}_{\mathcal{U}_{(n,p)}/A_n}.$$

We can see that $H^2_p(\mathcal{U}_{(n,p)}, f^*_{(n,p)}\Omega^1_{\operatorname{Spec} A_n/k} \otimes \omega_{\mathcal{U}_{(n,p)}/A_n})$ is a direct summand of

$$H^2_q(\mathcal{V}_{(n,p)}, g^*_{(n,p)}\Omega^1_{\operatorname{Spec} A_n/k}) \simeq H^2_q(\mathcal{V}_{(n-1,p)}, \mathcal{O}_{\mathcal{V}_{(n-1,p)}})$$

and this is zero by Claim 2.4(ii).

□

We define $o'_{\xi_n} \in \operatorname{Ext}^2(\iota_*(\Omega^1_U \otimes \omega_U), \omega_X)$ to be the element corresponding to the sequence (8). Let $r_2 \colon \operatorname{Ext}^2(\iota_*(\Omega^1_U \otimes \omega_U), \omega_X) \to \operatorname{Ext}^2(\Omega^1_U \otimes \omega_U, \omega_U)$ be the natural restriction map and $T \colon \operatorname{Ext}^2(\Omega^1_U \otimes \omega_U, \omega_U) \to \operatorname{Ext}^2(\Omega^1_U, \mathcal{O}_U)$ be the map induced by tensoring ω_U^{-1}. Then we have

$$T(r_2(o'_{\xi_n})) = o_{\xi_n}.$$

Hence it is enough to show that $\operatorname{Ext}^2(\iota_*(\Omega^1_U \otimes \omega_U), \omega_X) = 0$. By the Serre duality, we have $\operatorname{Ext}^2(\iota_*(\Omega^1_U \otimes \omega_U), \omega_X)^* \simeq H^1(X, \iota_*(\Omega^1_U \otimes \omega_U))$, where * is the dual.

In the following, we show that

$$H^1(X, \iota_*(\Omega^1_U \otimes \omega_U)) = 0.$$

Let m be a positive integer such that $-mK_X$ is very ample and $|-mK_X|$ contains a smooth member D_m which is disjoint with the singular points of X. Let $\pi_m \colon Y_m := \operatorname{Spec} \oplus_{i=0}^{m-1}\mathcal{O}_X(iK_X) \to X$ be a cyclic cover determined by D_m. Note that Y_m has only terminal Gorenstein singularities. There is an exact sequence

$$0 \to \iota_*(\Omega^1_U \otimes \omega_U) \to \iota_*(\Omega^1_U(\log D_m) \otimes \omega_U) \to \omega_X|_{D_m} \to 0$$

and it induces an exact sequence

$$H^0(X, \omega_X|_{D_m}) \to H^1(X, \iota_*(\Omega^1_U \otimes \omega_U)) \to H^1(X, \iota_*(\Omega^1_U(\log D_m) \otimes \omega_U)).$$

We have $H^0(X, \omega_X|_{D_m}) = 0$ since $-K_X$ is ample. Therefore, it is enough to show that

$$H^1(X, \iota_*(\Omega^1_U(\log D_m) \otimes \omega_U)) = 0.$$

Put $D' := \pi_m^{-1}(D_m)$ which satisfies that $D' \simeq D_m$ and $\pi_m^* D_m = mD'$. By using the isomorphism

$$(\pi_m)_* \left(\Omega^1_{Y_m}(\log D')(-D')\right) \simeq \bigoplus_{i=0}^{m-1} \iota_* \left(\Omega^1_U(\log D_m) \otimes \mathcal{O}_U\left((i+1) K_U\right)\right),$$

we can see that $H^1(X, \iota_*(\Omega^1_U(\log D_m) \otimes \omega_U))$ is a direct summand of

$$H^1(Y_m, \Omega^1_{Y_m}(\log D')(-D')).$$

We can show that

$$H^1(Y_m, \Omega^1_{Y_m}(\log D')(-D')) = 0$$

as follows. There is an exact sequence

$$0 \to \Omega^1_{Y_m}(\log D')(-D') \to \Omega^1_{Y_m} \to \Omega^1_{D'} \to 0$$

and it induces an exact sequence

$$H^0(D', \Omega^1_{D'}) \to H^1(Y_m, \Omega^1_{Y_m}(\log D')(-D')) \to H^1(Y_m, \Omega^1_{Y_m}) \xrightarrow{\beta} H^1(D', \Omega^1_{D'}).$$

We can see that $H^1(D', \mathcal{O}_{D'}) = 0$ since $D_m \simeq D'$ and we have an exact sequence

$$0 \to \mathcal{O}_X(-D_m) \to \mathcal{O}_X \to \mathcal{O}_{D_m} \to 0.$$

This and the Hodge symmetry imply $H^0(D', \Omega^1_{D'}) = 0$. Hence it is enough to show that β is injective. We use the following commutative diagram

$$\begin{tikzcd}
H^1(Y_m, \Omega^1_{Y_m}) \arrow[r, "\beta"] & H^1(D', \Omega^1_{D'}) \\
H^1(Y_m, \mathcal{O}^*_{Y_m}) \otimes \mathbb{C} \arrow[r, "\gamma"] \arrow[u, "\phi"] \arrow[d, "\beta_1"] & H^1(D', \mathcal{O}^*_{D'}) \otimes \mathbb{C} \arrow[u, "\psi"'] \arrow[d, "\beta_2"'] \\
H^2(Y_m, \mathbb{C}) \arrow[r, "\delta"] & H^2(D', \mathbb{C}).
\end{tikzcd}$$

We can see that δ is injective by Theorem 2.6 since Y_m has only l.c.i. singularities. Note that β_1 is an isomorphism since $H^i(Y_m, \mathcal{O}_{Y_m}) = 0$ for $i = 1, 2$. Hence $\delta \circ \beta_1 = \beta_2 \circ \gamma$ is injective. This implies that γ is injective. We can show that ϕ is surjective by an argument which is similar to that in [17](2.2). Note that ψ is injective since D' is a smooth surface and $H^1(D', \mathcal{O}_{D'}) = 0$. Hence $\psi \circ \gamma = \beta \circ \phi$ is injective. Therefore β is injective.

Hence we proved $o_{\xi_n} = 0$. It is enough for unobstructedness by Proposition 2.2 since X is a projective variety and has a semi-universal deformation space. □

Remark 2.9. For a Fano 3-fold X with canonical singularities, its Kuranishi space $\mathrm{Def}(X)$ is not smooth in general. For example, let X be a cone over the del Pezzo surface of degree 6. Then X has 2 different smoothings.

Next, we study deformations of a $\mathbb{Q}$-Fano 3-fold with its pluri-anticanonical element.

Theorem 2.10. *Let X be a $\mathbb{Q}$-Fano 3-fold and m a positive integer. Assume that $|-mK_X|$ contains an element D. Let $\mathrm{Def}(X, D)$ and $\mathrm{Def}(X)$ be the Kuranishi space of the pair (X, D) and the Kuranishi space of X respectively.*

Then the forgetting map $\mathrm{Def}(X, D) \to \mathrm{Def}(X)$ is a smooth morphism. In particular, the deformations of the pair (X, D) are unobstructed.

Proof. Set $k := \mathbb{C}$. Let A be an Artin local k-algebra, $e = (0 \to k \to \tilde{A} \to A \to 0)$ a small extension and $\zeta := (f\colon (\mathcal{X}, \mathcal{D}) \to \operatorname{Spec} A)$ a flat deformation of the pair (X, D). Assume that we have a lifting $\tilde{\mathcal{X}} \to \operatorname{Spec} \tilde{A}$ of $f\colon \mathcal{X} \to \operatorname{Spec} A$. It is enough to show that there exists a lifting $\tilde{\mathcal{D}} \subset \tilde{\mathcal{X}}$ of $\mathcal{D} \subset \mathcal{X}$. Let $\mathcal{N}_{D/X}$ be the normal sheaf of $D \subset X$. Since an obstruction to the existence of such a lifting lies in $H^1(D, \mathcal{N}_{D/X})$, it is enough to show that

$$H^1(D, \mathcal{N}_{D/X}) = 0.$$

There is an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{N}_{D/X} \to 0$$

and this induces an exact sequence

$$H^1(X, \mathcal{O}_X(D)) \to H^1(D, \mathcal{N}_{D/X}) \to H^2(X, \mathcal{O}_X).$$

The L.H.S and R.H.S. are zero by the Kodaira vanishing theorem. Hence we have $H^1(D, \mathcal{N}_{D/X}) = 0$. □

3. A $\mathbb{Q}$-smoothing of a $\mathbb{Q}$-Fano 3-fold: the ordinary case

3.1. **Stratification on the Kuranishi space of a singularity.** First, we recall a stratification on the Kuranishi space of an isolated singularity introduced in the proof of Theorem 2.4 [19].

Let V be a Stein space with an isolated hypersurface singularity $p \in V$. Then we have its semi-universal deformation space $\operatorname{Def}(V)$ and the semi-universal family $\mathcal{V} \to \operatorname{Def}(V)$. It has a stratification into Zariski locally closed and smooth subsets $S_k \subset \operatorname{Def}(V)$ for $k \geq 0$ with the following properties;

- $\operatorname{Def}(V) = \amalg_{k \geq 0} S_k$.
- S_0 is a non-empty Zariski open subset of $\operatorname{Def}(V)$ and $\mathcal{V}$ is smooth over S_0.
- S_k are of pure codimension in $\operatorname{Def}(V)$ for all $k > 0$ and $\operatorname{codim}_{\operatorname{Def}(V)} S_k < \operatorname{codim}_{\operatorname{Def}(V)} S_{k+1}$.
- If $k > l$, then $\overline{S_k} \cap S_l = \emptyset$.
- $\mathcal{V}$ has a simultaneous resolution on each S_k, that is, there is a resolution of $\mathcal{V} \times_{\operatorname{Def}(V)} S_k$ which is smooth over S_k.

3.2. **A useful homomorphism between cohomology groups.** Let us explain the homomorphism which we need for finding $\mathbb{Q}$-smoothings. Let $p \in U$ be a 3-dimensional Stein neighbourhood of a terminal singularity p of index r. Fix a positive integer m such that $r | m$. Let

$$\pi_U\colon V := \operatorname{Spec} \bigoplus_{i=0}^{m-1} \mathcal{O}(iK_U) \to U$$

be a finite morphism defined by the isomorphism $\mathcal{O}_U(rK_U) \simeq \mathcal{O}_U$. Note that V is a disjoint union of several copies of the index 1 cover of U. Let $G := \mathbb{Z}/m\mathbb{Z}$ be the Galois group of π_U. Set $Q := \pi_U^{-1}(p)$. Let $\nu_V\colon \tilde{V} \to V$ be a G-equivariant good resolution, $F_V := \nu_V^{-1}(Q) = \operatorname{Exc}(\nu_V)$ its exceptional locus which has normal crossing support and $\tilde{U} := \tilde{V}/G$ its quotient. So we have a diagram

(9)
$$\begin{tikzcd} \tilde{V} \arrow[r, "\tilde{\pi}_U"] \arrow[d, "\nu_V"] & \tilde{U} \arrow[d, "\mu_U"] \\ V \arrow[r, "\pi_U"] & U. \end{tikzcd}$$

Let $\mathcal{F}_U^{(0)}$ be the $\mathbb{Z}_m$-invariant part of $(\tilde{\pi}_U)_*(\Omega^2_{\tilde{V}}(\log F_V)(-F_V - \nu_V^* K_V))$. Set $V' := V \setminus Q$. We have a coboundary map of the local cohomology group

$$\tau_V \colon H^1(V', \Omega^2_{V'} \otimes \omega_{V'}^{-1}) \to H^2_{F_V}(\tilde{V}, \Omega^2_{\tilde{V}}(\log F_V)(-F_V - \nu_V^* K_V)).$$

Its G-invariant part is

$$\phi_U \colon H^1(U', \Omega^2_{U'} \otimes \omega_{U'}^{-1}) \to H^2_{E_U}(\tilde{U}, \mathcal{F}_U^{(0)}),$$

where $U' := U \setminus \{p\}$ is the punctured neighbourhood and $E_U \subset \tilde{U}$ is the exceptional locus of μ_U.

If ϕ_U is nonzero, it is useful for finding a good deformation direction. The following is a treatable case.

Proposition 3.1. *Let $p \in U$ be a Stein neighbourhood as above such that V is not smooth. Assume that $p \in U$ is ordinary, that is, the defining equation of V is G-invariant* ([13] Definition 2.5(2))*. Then $\phi_U \neq 0$.*

Proof. By [19] Theorem 1.1, $\operatorname{Ker} \tau_V \subset H^1(V', \Omega^2_{V'} \otimes \omega_{V'}^{-1})$ is a proper $\mathcal{O}_{V,Q}$-submodule. There exists a G-invariant good direction ([13, Definition 2.3])

$$\eta \in H^1(V', \Omega^2_{V'} \otimes \omega_{V'}^{-1}) \setminus \operatorname{Ker} \tau_V$$

by Corollary 2.6 in [13]. Then we have the corresponding element $\eta \in H^1(U', \Omega^2_{U'} \otimes \omega_{U'}^{-1}) \setminus \operatorname{Ker} \phi_U$. □

3.3. **Proof of the theorem.** We can find good first order deformations as follows.

Theorem 3.2. *Let X be a $\mathbb{Q}$-Fano 3-fold.*

Then X has a deformation $f \colon \mathcal{X} \to \Delta^1$ over an unit disc such that the singularities on $\mathcal{X}_t$ for $t \neq 0$ satisfy the following condition;

Let $p_t \in \mathcal{X}_t$ be a singular point and U_{p_t} its Stein neighbourhood. Then $\phi_{U_{p_t}} = 0$, where $\phi_{U_{p_t}}$ is the homomorphism defined in Section 3.2.

Proof. Let $p_1, \ldots, p_l \in X$ be non-rigid singular points of X such that $p_1, \ldots, p_{l'}$ for some $l' \leq l$ are the points which satisfy

$$\phi_{U_i} \neq 0$$

for $i = 1, \ldots, l'$, where U_i is a small Stein neighbourhood of p_i.

Let m be a sufficiently large integer such that $-mK_X$ is very ample and $|-mK_X|$ contains a smooth member D_m such that $D_m \cap \operatorname{Sing} X = \emptyset$. Let

$$\pi \colon Y := \operatorname{Spec} \bigoplus_{i=0}^{m-1} \mathcal{O}_X(iK_X) \to X$$

be a cyclic cover determined by D_m. There exists a good $\mathbb{Z}_m$-equivariant resolution ([1]) $\nu \colon \tilde{Y} \to Y$ which induces an isomorphism $\nu^{-1}(Y \setminus \pi^{-1}\{p_1, \ldots, p_l\}) \to Y \setminus \pi^{-1}\{p_1, \ldots, p_l\}$

and a birational morphism $\mu\colon \tilde{X} := \tilde{Y}/\mathbb{Z}_m \to X$. These induce the following cartesian diagram;

$$\begin{tikzcd} \tilde{Y} \arrow[r, "\tilde{\pi}"] \arrow[d, "\nu"] & \tilde{X} \arrow[d, "\mu"] \\ Y \arrow[r, "\pi"] & X. \end{tikzcd} \tag{10}$$

Let $\pi_i\colon V_i := \pi^{-1}(U_i) \to U_i$ and $\nu_i\colon \tilde{V}_i := \nu^{-1}(V_i) \to V_i$ be morphisms induced by the morphisms in the above diagram. Put $\tilde{U}_i := \tilde{V}_i/\mathbb{Z}_m$. Then we get the following cartesian diagram;

$$\begin{tikzcd} \tilde{V}_i \arrow[r, "\tilde{\pi}_i"] \arrow[d, "\nu_i"] & \tilde{U}_i \arrow[d, "\mu_i"] \\ V_i \arrow[r, "\pi_i"] & U_i. \end{tikzcd} \tag{11}$$

Put $F := \operatorname{Exc}(\nu), E := \operatorname{Exc}(\mu), D' := \pi^{-1}(D_m)$ and $L' := \mathcal{O}_Y(D') = \mathcal{O}_Y(\pi^*(-K_X))$. Note that F has normal crossing support since ν is good. Also put $F_i := \operatorname{Exc}(\nu_i)$ and $E_i := \operatorname{Exc}(\mu_i)$. Let $\mathcal{F}^{(0)}$ be the $\mathbb{Z}_m$-invariant part of $\tilde{\pi}_*(\Omega^2_{\tilde{Y}}(\log F)(-F) \otimes \nu^* L')$. Let U be the smooth part of X. Note that $\mathcal{F}^{(0)}|_U \simeq \Omega^2_U \otimes \omega_U^{-1}$. Set $\mathcal{F}^{(0)}_i := \mathcal{F}^{(0)}|_{\tilde{U}_i}$ and $U'_i := U_i \setminus \{p_i\}$. Note that $\mathcal{F}^{(0)}_i|_{U'_i} \simeq \Omega^2_{U'_i} \otimes \omega^{-1}_{U'_i}$.

We have the following commutative diagram;

$$\begin{tikzcd} H^1(U, \Omega^2_U \otimes \omega_U^{-1}) \arrow[r, "\oplus\psi_i"] \arrow[d, "\oplus p_{U_i}"] & \oplus_{i=1}^{l'} H^2_{E_i}(\tilde{X}, \mathcal{F}^{(0)}) \arrow[r] \arrow[d, "\simeq"] & H^2(\tilde{X}, \mathcal{F}^{(0)}) \\ \oplus_{i=1}^{l'} H^1(U'_i, \Omega^2_{U'_i} \otimes \omega^{-1}_{U'_i}) \arrow[r, "\oplus\phi_i"] & \oplus_{i=1}^{l'} H^2_{E_i}(\tilde{U}_i, \mathcal{F}^{(0)}_i). & \end{tikzcd} \tag{12}$$

We identify $H^2_{E_i}(\tilde{X}, \mathcal{F}^{(0)})$ and $H^2_{E_i}(\tilde{U}_i, \mathcal{F}^{(0)}_i)$ by the natural homomorphism induced by restriction. Note that $\mathcal{F}^{(0)}_i \simeq \mathcal{F}^{(0)}_{U_i}$, where $\mathcal{F}^{(0)}_{U_i}$ is the sheaf defined in Section 3.2. Hence ϕ_i is ϕ_{U_i} in Section 3.2.

Let $T^1_X, T^1_{V_i}, T^1_{U_i}$ be the tangent spaces of $\operatorname{Def}(X), \operatorname{Def}(V_i), \operatorname{Def}(U_i)$ respectively. By [26] §1 Theorem 2 or the proof of Proposition 2.3 in this paper, we can see that the first order deformations of V_i, U_i are bijective to those of the smooth part V'_i, U'_i. Similarly we can see the same correspondence for X. So we have

$$T^1_X \simeq H^1(U, \Theta_U) \simeq H^1(U, \Omega^2_U \otimes \omega_U^{-1}),$$

$$T^1_{V_i} \simeq H^1(V'_i, \Theta_{V'_i}) \simeq H^1(V'_i, \Omega^2_{V'_i} \otimes \omega^{-1}_{V'_i}),$$

$$T^1_{U_i} \simeq H^1(U'_i, \Theta_{U'_i}) \simeq H^1(U'_i, \Omega^2_{U'_i} \otimes \omega^{-1}_{U'_i}),$$

where $\Theta_U, \Theta_{V'_i}, \Theta_{U'_i}$ are the tangent sheaves of U, V'_i, U'_i respectively. Hence p_{U_i} is regarded as the restriction homomorphism $T^1_X \to T^1_{U_i}$.

Since $\tilde{\pi}$ is finite, $H^2(\tilde{X}, \mathcal{F}^{(0)})$ is a direct summand of

$$H^2(\tilde{Y}, \Omega^2_{\tilde{Y}}(\log F)(-F) \otimes \nu^* L')$$

and this is zero by the vanishing theorem by Guillen-Navarro Aznar-Puerta-Steenbrink ([20] Theorem 7.30 (a)). Hence $\oplus\psi_i$ is surjective.

By the assumption that $\phi_i \neq 0$ for $i = 1, \dots, l'$, there exists $\eta_i \in H^1(U'_i, \Omega^2_{U'_i} \otimes \omega^{-1}_{U'_i}) \setminus \operatorname{Ker} \phi_i$. By the surjectivity of $\oplus\psi_i$, there exists $\eta \in H^1(U, \Omega^2_U \otimes \omega_U^{-1})$ such that $\psi_i(\eta) = \phi_i(\eta_i)$. Then we have $p_{U_i}(\eta) \notin \operatorname{Ker}(\phi_i)$.

Since V_i has only rational singularities, the birational morphism $\nu_i \colon \tilde{V}_i \to V_i$ induces a morphism $\operatorname{Def}(\tilde{V}_i) \to \operatorname{Def}(V_i)$ of their Kuranishi spaces and the homomorphism $H^1(\tilde{V}_i, \Theta_{\tilde{V}_i}) \to H^1(V'_i, \Theta_{V'_i})$ on their tangent spaces. This homomorphism can be rewritten as

$$(\nu_i)_* \colon H^1(\tilde{V}_i, \Omega^2_{\tilde{V}_i} \otimes \omega^{-1}_{\tilde{V}_i}) \to H^1(V'_i, \Omega^2_{V'_i} \otimes \omega^{-1}_{V'_i})$$

and this is a homomorphism induced by an open immersion $V'_i \hookrightarrow \tilde{V}_i$. Note that infinitesimal deformations of U_i come from $\mathbb{Z}_m$-equivariant deformations of V_i and $H^1(U'_i, \Theta_{U'_i}) \simeq H^1(V'_i, \Theta_{V'_i})^{\mathbb{Z}_m}$.

Note that ϕ_i is the $\mathbb{Z}_m$-invariant part of the homomorphism

$$\tau_i \colon H^1(V'_i, \Omega^2_{V'_i} \otimes \omega^{-1}_{V'_i}) \to H^2_{F_i}(\tilde{V}_i, \Omega^2_{\tilde{V}_i}(\log F_i)(-F_i - \nu_i^* K_{V_i})).$$

Claim 3.3. $\operatorname{Im}(\nu_i)_* \subset \operatorname{Ker} \tau_i$.

Proof of Claim. We can write

$$K_{\tilde{V}_i} = \nu_i^* K_{V_i} + \sum_{j=1}^{m_i} a_{i,j} F_{i,j},$$

where $F_i = \bigcup_{j=1}^{m_i} F_{i,j}$ is the irreducible decomposition and $a_{i,j} \geq 1$ are some integers for $j = 1, \dots, m_i$ since V_i is terminal Gorenstein. We can define a homomorphism

$$\alpha_i \colon H^1(\tilde{V}_i, \Omega^2_{\tilde{V}_i} \otimes \omega^{-1}_{\tilde{V}_i}) \to H^1(\tilde{V}_i, \Omega^2_{\tilde{V}_i}(\log F_i)(-F_i - \nu_i^* K_{V_i}))$$

as a composite of the following homomorphisms;

$$\begin{aligned} &\alpha_i \colon H^1(\tilde{V}_i, \Omega^2_{\tilde{V}_i} \otimes \omega^{-1}_{\tilde{V}_i}) = H^1(\tilde{V}_i, \Omega^2_{\tilde{V}_i}(-\sum_{j=1}^{m_i} a_{i,j} F_{i,j} - \nu_i^* K_{V_i})) \\ &\to H^1(\tilde{V}_i, \Omega^2_{\tilde{V}_i}(\log F_i)(-\sum_{j=1}^{m_i} a_{i,j} F_{i,j} - \nu_i^* K_{V_i})) \to H^1(\tilde{V}_i, \Omega^2_{\tilde{V}_i}(\log F_i)(-F_i - \nu_i^* K_{V_i})) \end{aligned} \tag{13}$$

since $a_{i,j} \geq 1$.

Note that $\operatorname{Ker} \tau_i = \operatorname{Im} \rho_i$, where we put

$$\rho_i \colon H^1(\tilde{V}_i, \Omega^2_{\tilde{V}_i}(\log F_i)(-F_i - \nu_i^* K_{V_i})) \to H^1(V'_i, \Omega^2_{V'_i} \otimes \omega^{-1}_{V'_i}).$$

We can see that $(\nu_i)_*$ factors as

$$(\nu_i)_* \colon H^1(\tilde{V}_i, \Omega^2_{\tilde{V}_i} \otimes \omega^{-1}_{\tilde{V}_i}) \xrightarrow{\alpha_i} H^1(\tilde{V}_i, \Omega^2_{\tilde{V}_i}(\log F_i)(-F_i - \nu_i^* K_{V_i})) \xrightarrow{\rho_i} H^1(V'_i, \Omega^2_{V'_i} \otimes \omega^{-1}_{V'_i}).$$

Hence $\operatorname{Ker} \tau_i = \operatorname{Im} \rho_i \supset \operatorname{Im}(\nu_i)_*$. □

Hence we get $p_{U_i}(\eta) \notin \operatorname{Im}(\nu_i)_*$. Let r_i be the Gorenstein index of the singular point p_i and $\pi_i^{-1}(p_i) =: \{q_{i1}, \dots, q_{ik(i)}\}$, where $k(i) := \frac{m}{r_i}$. Let

$$V_i := \amalg_{j=1}^{k(i)} V_{i,j}$$

be the decomposition into the connected components of V_i. Fix a stratification on each $\mathrm{Def}(V_{i,j})$ for $j = 1, \dots, k(i)$ as in Section 3.1. Let $g\colon \mathcal{X} \to \Delta^1$ be a small deformation of X over a disc determined by $\eta \in H^1(U, \Theta_U)$. This determines a holomorphic map $\varphi_i\colon \Delta^1 \to \mathrm{Def}(U_i)$. By considering the index 1 cover, we can see that φ_i induces a holomorphic map $\varphi_{i,1}\colon \Delta^1 \to \mathrm{Def}(V_{i,1})$. Let $S_{i,k}$ be the minimal stratum of $\mathrm{Def}(V_{i,1})$. Then the image of $\varphi_{i,1}$ is not contained in $S_{i,k}$. and, for general $t \in \Delta^1$, we have $\varphi_{i,1}(t) \in S_{i,k'}$ for some $k' < k$. We can continue this process as long as $\phi_i \neq 0$ and reach a $\mathbb{Q}$-Fano 3-fold with the required property. □

Proposition 3.1 and Theorem 3.2 imply the following.

Corollary 3.4. *Let X be a $\mathbb{Q}$-Fano 3-fold with only ordinary terminal singularities. Then X has a $\mathbb{Q}$-smoothing.*

Proof. By Proposition 3.1, we can continue the process in the proof of Theorem 3.2 until we get a $\mathbb{Q}$-smoothing since deformations of ordinary terminal singularities are ordinary. □

Remark 3.5. The author does not know ϕ_U is zero or not when U is a Stein neighbourhood of an exceptional terminal singularity. If we can prove $\phi_U \neq 0$ in that case, it implies Conjecture 1.5 by the above proof of Theorem 3.2.

Remark 3.6. There is an example of a weak Fano 3-fold which does not have a smoothing. It is written in [14, Example 3.7].

4. A $\mathbb{Q}$-smoothing of a $\mathbb{Q}$-Fano 3-fold with a Du Val elephant

4.1. **Existence of an essential resolution of a pair.** We need a resolution of a 3-dimensional hypersurface singularity and its divisor with good properties as follows.

Proposition 4.1. *Let Y be a 3-dimensional variety with only hypersurface singularities over an algebraically closed field k of characteristic 0. Let $D \subset Y$ be a reduced Cartier divisor. Then the following statements hold.*

There exists a functorial resolution of singularities $f\colon X \to Y$ of the pair (Y, D) which is a composite of blow-ups along smooth centers such that integers a_j in the equality

$$((K_X + \tilde{D}) - f^*(K_Y + D))|_{U_{\tilde{D}}} = \sum a_j E_j|_{U_{\tilde{D}}} \tag{14}$$

satisfies $a_j \leq 0$ for some open neighbourhood $U_{\tilde{D}} \subset X$ of $\tilde{D}$ and $\tilde{D}$ is smooth, where $\tilde{D} \subset X$ is the strict transform of D and $E_j \subset X$ are irreducible f-exceptional divisors.
If Y is smooth, we can take $U_{\tilde{D}} = X$.

Remark 4.2. If we restrict the relation (14) to $\tilde{D}$, we see that $f_D^* K_D - K_{\tilde{D}}$ is an effective divisor supported on $\mathrm{Exc}\, f_D$ where $f_D\colon \tilde{D} \to D$ is induced by f. Such a resolution is called an *essential* resolution of singularities in [9] Definition 3.5 although we do not assume goodness of resolution.

Proof. We can assume that $Y \subset W$ is a closed affine variety defined by some function $h \in H^0(W, \mathcal{O}_W)$ in some smooth affine 4-fold W and there exists a divisor $A \subset W$ such that $D = A \cap Y$. Same proof in the following works for general Y and D.

There exists a smooth surface Δ_m and a functorial resolution

$$f_{\Delta_1}\colon \Delta_m \overset{\mu_{m-1}}{\to} \Delta_{m-1} \to \cdots \to \Delta_2 \overset{\mu_1}{\to} \Delta_1 := D$$

of D, where Δ_i are singular for $i = 1, \dots, m-1$ and $\mu_i \colon \Delta_{i+1} \to \Delta_i$ is a blow-up of a smooth subvariety $Z_i \subset \operatorname{Sing} \Delta_i$ (cf. [8]). Let

$$f_{Y_1} \colon Y_m \overset{\nu_{m-1}}{\to} Y_{m-1} \to \cdots \to Y_2 \overset{\nu_1}{\to} Y_1 := Y,$$

$$f_{W_1} \colon W_m \overset{\rho_{m-1}}{\to} W_{m-1} \to \cdots \to W_2 \overset{\rho_1}{\to} W_1 := W$$

be birational morphisms, where $\nu_i \colon Y_{i+1} \to Y_i, \rho_i \colon W_{i+1} \to W_i$ are blow-ups of Z_i. Let $G_i \subset W_m$ and $E_i \subset Y_m$ be the strict transforms of $\rho_i^{-1}(Z_i)$ and $\nu_i^{-1}(Z_i)$ respectively. Let $f_{W_i} \colon W_m \to W_i, f_{Y_i} \colon Y_m \to Y_i$ be composites of the above blow-ups.

Let $A_i \subset W_i$ be the strict transform of A. Note that Y_m is smooth around Δ_m since Δ_m is a smooth Cartier divisor.

Then we can write

$$K_{Y_m} + \Delta_m = f_{Y_1}^*(K_{Y_1} + \Delta_1) + F \tag{15}$$

for some divisor F supported in $\operatorname{Exc} f_{Y_1}$.

Claim 4.3. $-F$ is effective.

Proof of Claim. We can write

$$K_{Y_m} + \Delta_m - f_{Y_1}^*(K_{Y_1} + \Delta_1) = \sum_{i=1}^{m-1} f_{Y_{i+1}}^*(K_{Y_{i+1}} + \Delta_{i+1} - \nu_i^*(K_{Y_i} + \Delta_i)) \tag{16}$$

and $K_{Y_{i+1}} + \Delta_{i+1} - \nu_i^*(K_{Y_i} + \Delta_i) = F_i$, where F_i is a divisor on Y_{i+1} supported on $\operatorname{Exc} \nu_i$. It is enough to show that $-F_i$ is effective.

First we consider the case where Y_i is smooth around Z_i. We can see that the coefficient b_i of $\nu_i^{-1}(Z_i)$ in $K_{Y_{i+1}} + \Delta_{i+1} - \nu_i^*(K_{Y_i} + \Delta_i)$ is

$$b_i = \operatorname{codim}_{Y_i} Z_i - 1 - \operatorname{mult}_{Z_i} \Delta_i.$$

We see that $b_i \leq 0$ since $Z_i \subset \operatorname{Sing} \Delta_i$ implies that $\operatorname{mult}_{Z_i} \Delta_i \geq 2$. Hence $-F_i$ is effective.

Next consider the case where Y_i has a singularity on Z_i. We see that

$$(K_{W_{i+1}} + A_{i+1} + Y_{i+1} - \rho_i^*(K_{W_i} + A_i + Y_i))|_{Y_{i+1}} = K_{Y_{i+1}} + \Delta_{i+1} - \nu_i^*(K_{Y_i} + \Delta_i)$$

and that the coefficient of $\rho_i^{-1}(Z_i)$ in

$$K_{W_{i+1}} + A_{i+1} + Y_{i+1} - \rho_i^*(K_{W_i} + A_i + Y_i)$$

is

$$\operatorname{codim}_{Y_i} Z_i - \operatorname{mult}_{Z_i} A_i - \operatorname{mult}_{Z_i} Y_i$$

and this is non-positive since $\operatorname{mult}_{Z_i} Y_i \geq 2$ if $\dim Z_i = 0$. Hence we see that $-F_i$ is effective. $\square$

If Y is smooth, then Y_m is smooth and we are done. Note that we do not need to take an open neighbourhood $U_{\tilde{D}}$ in this case.

If Y is not smooth, then Y_m is smooth around Δ_m. Hence we get a required birational morphism by blowing up smooth centres outside Δ_m. $\square$

4.2. **Classification of 3-dimensional terminal singularities.** Let $(p \in U)$ be a germ of a 3-dimensional terminal singularity. By Reid's result [25], (U, p) is locally isomorphic to

$$0 \in (f = 0)/\mathbb{Z}_r \subset \mathbb{C}^4/\mathbb{Z}_r,$$

where $\mathbb{Z}_r$ acts on $\mathbb{C}^4$ diagonally and $f \in \mathbb{C}[x, y, z, u]$ and x, y, z, u are $\mathbb{Z}_r$-semi-invariant functions on $\mathbb{C}^4$. By the list in [25](6.4), we have a $\mathbb{Z}_r$-semi-invariant function $h \in \mathbb{C}[x, y, z, u]$ such that

$$D_h := (f = h = 0)/\mathbb{Z}_r \subset (f = 0)/\mathbb{Z}_r =: U_f$$

has only a Du Val singularity at the origin and $D_h \in |-K_{U_f}|$.

4.3. **Proof of the theorem.**

Proof of Theorem 1.9. By Corollary 3.4, we can assume that the singularities on X are non ordinary terminal singularities. Let m be a positive integer such that $-mK_X$ is very ample and $|-mK_X|$ contains a smooth element D_m which satisfies $D_m \cap \operatorname{Sing} D = \emptyset$ and intersects transversely with D. Let $\pi\colon Y := \operatorname{Spec} \oplus_{i=0}^{m-1} \mathcal{O}_X(iK_X) \to X$ be a cyclic cover determined by D_m. Note that Y is terminal Gorenstein. Put $\{p_1, \ldots, p_l\} := \operatorname{Sing} D$. Note that $\operatorname{Sing} X \subset \operatorname{Sing} D$ since all the singularities on X are non-Gorenstein. Also note that $G := \operatorname{Gal}(Y/X) \simeq \mathbb{Z}_m$ acts on Y and $\Delta := \pi^{-1}(D)$ is G-invariant.

Let U_i be a sufficiently small affine neighbourhood of p_i such that $U_i \setminus \{p_i\}$ is smooth and $K_{V_i} = 0$, where $V_i := \pi^{-1}(U_i)$. Let $\pi_i : V_i \to U_i$ be the morphism induced by π.

By Proposition 4.1, we can take a $\mathbb{Z}_m$-equivariant resolution $\nu\colon \tilde{Y} \to Y$ of Y such that $\nu|_{\nu^{-1}(Y \setminus \operatorname{Sing}\Delta)}$ is an isomorphism, $\tilde{\Delta} := (\nu^{-1})_*\Delta$ is smooth and

$$\nu_\Delta^* K_\Delta = K_{\tilde{\Delta}},$$

where $\nu_\Delta\colon \tilde{\Delta} \to \Delta$ is induced by ν. Then we have the following diagram;

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\tilde{\pi}} & \tilde{X} \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \mu} \\ Y & \xrightarrow{\pi} & X. \end{array} \tag{17}$$

We also have the following diagram induced by the above diagram;

$$\begin{array}{ccc} \tilde{V}_i & \xrightarrow{\tilde{\pi}_i} & \tilde{U}_i \\ \downarrow{\scriptstyle \nu_i} & & \downarrow{\scriptstyle \mu_i} \\ V_i & \xrightarrow{\pi_i} & U_i. \end{array} \tag{18}$$

Put $F := \operatorname{Exc}(\nu) \subset \tilde{Y}$, $F_i := \operatorname{Exc}(\nu_i)$, $E := \operatorname{Exc}(\mu)$ and $E_i := \operatorname{Exc}(\mu_i)$. Put $\tilde{\Delta}_i := (\nu_i^{-1})_*\Delta_i$, where $\Delta_i := \Delta \cap V_i$.

Let $\mathcal{F}^{(0)}$ be the $\mathbb{Z}_m$-invariant part of $\tilde{\pi}_*\Omega^2_{\tilde{Y}}(\log \tilde{\Delta})$ and set $\mathcal{F}_i^{(0)} := \mathcal{F}^{(0)}|_{\tilde{U}_i}$. Set $U := X \setminus \operatorname{Sing} D$. Note that $\mathcal{F}^{(0)}|_U \simeq \Omega^2_U(\log D_U)$, where $D_U := D \cap U$.

Hence we have the following diagram;

$$\begin{array}{ccccc} H^1(U, \Omega^2_U(\log D_U)) & \xrightarrow{\oplus\psi_i} & \bigoplus_{i=1}^l H^2_{E_i}(\tilde{X}, \mathcal{F}^{(0)}) & \xrightarrow{\oplus\beta_i} & H^2(\tilde{X}, \mathcal{F}^{(0)}) \\ \downarrow{\scriptstyle \oplus p_{U_i}} & & \downarrow{\scriptstyle \simeq} & & \\ \bigoplus_{i=1}^l H^1(U_i', \Omega^2_{U_i'}(\log D_i')) & \xrightarrow{\oplus\phi_i} & \bigoplus_{i=1}^l H^2_{E_i}(\tilde{U}_i, \mathcal{F}_i^{(0)}), & & \end{array} \tag{19}$$

where $U_i' := U_i \setminus \{p_i\}$ and $D_i' := D \cap U_i'$. We see that

$$H^1(U, \Omega^2_U(\log D_U)) \simeq T^1_{(X,D)},$$

$$H^1(U_i', \Omega^2_{U_i'}(\log D_i')) \simeq T^1_{(U_i,D_i)},$$

by [10] Theorem 12 , where $T^1_{(X,D)}, T^1_{(U_i,D_i)}$ are the sets of the first order deformations of the pairs (X, D) and (U_i, D_i) respectively.

$\nu_i \colon \tilde{V}_i \to V_i$ induces a birational morphism $\nu_{\Delta_i} \colon \tilde{\Delta}_i \to \Delta_i$. Since Δ has only rational singularities, this induces a morphism $\mathrm{Def}(\tilde{\Delta}_i) \to \mathrm{Def}(\Delta_i)$ between the Kuranishi spaces of $\tilde{\Delta}_i, \Delta_i$ respectively ([31]). Let $T^1_{\tilde{\Delta}_i}, T^1_{\Delta_i}$ be the tangent spaces of $\mathrm{Def}(\tilde{\Delta}_i), \mathrm{Def}(\Delta_i)$ respectively. Then ν_{Δ_i} induces a linear map $(\nu_{\Delta_i})_* \colon T^1_{\tilde{\Delta}_i} \to T^1_{\Delta_i}$. It is well known that (cf. [3] 2.10) $(\nu_{\Delta_i})_* = 0$. By the construction of ν, we have

$$\nu^*_{\Delta_i} K_{\Delta_i} = K_{\tilde{\Delta}_i}. \tag{20}$$

Let $T^1_{(V_i,\Delta_i)}$ be the set of first order deformations of the pair (V_i, Δ_i). Note that

$$T^1_{(V_i,\Delta_i)} \simeq H^1(V_i', \Theta_{V_i'}(-\log \Delta_i')) \simeq H^1(V_i', \Omega^2_{V_i'}(\log \Delta_i')(-K_{V_i'} - \Delta_i')) \simeq H^1(V_i', \Omega^2_{V_i'}(\log \Delta_i')),$$

where $V_i' := V_i \setminus \pi^{-1}(p_i)$ and $\Delta_i' := \Delta_i \setminus \pi^{-1}(p_i)$ since V_i is affine, $K_{V_i} = 0$, $\Delta_i \in |-K_{V_i}|$ and [18] Lemma 1. Also note that ϕ_i is the $\mathbb{Z}_m$-invariant part of the coboundary map

$$\tau_i \colon H^1(V_i', \Omega^2_{V_i'}(\log \Delta_i')) \to H^2_{F_i}(\tilde{V}_i, \Omega^2_{\tilde{V}_i}(\log \tilde{\Delta}_i)).$$

Claim 4.4. $p_{\Delta_i}(\operatorname{Ker} \tau_i) \subset \operatorname{Im}(\nu_{\Delta_i})_* = 0$, where we use the forgetting homomorphism

$$p_{\Delta_i} \colon H^1(V_i', \Omega^2_{V_i'}(\log \Delta_i')) \simeq T^1_{(V_i,\Delta_i)} \to T^1_{\Delta_i}. \tag{21}$$

Proof of Claim. We have the following commutative diagram

$$\begin{tikzcd}
H^1(\tilde{V}_i, \Omega^2_{\tilde{V}_i}(\log \tilde{\Delta}_i)) \arrow[r, "\alpha_i"] \arrow[d] & H^1(V_i', \Omega^2_{V_i'}(\log \Delta_i')) \arrow[d, "\simeq"] \\
H^1(\tilde{\Delta}_i, \Omega^1_{\tilde{\Delta}_i}) \arrow[d, "\simeq"] & H^1(V_i', \Omega^2_{V_i'}(\log \Delta_i')(-K_{V_i'} - \Delta_i')) \arrow[d, "p_{\Delta_i}"] \\
H^1(\tilde{\Delta}_i, \Omega^1_{\tilde{\Delta}_i}(-K_{\tilde{\Delta}_i})) \arrow[dr, "\simeq"] & T^1_{\Delta_i} \\
 & T^1_{\tilde{\Delta}_i} \arrow[u, "(\nu_{\Delta_i})_*"']
\end{tikzcd} \tag{22}$$

by the relation (20). By this diagram, we can see that $p_{\Delta_i}(\operatorname{Ker} \tau_i) = \operatorname{Im}(p_{\Delta_i} \circ \alpha_i) \subset \operatorname{Im}(\nu_{\Delta_i})_*$. $\square$

There exists $\eta_i \in T^1_{(U_i,D_i)}$ which induces a simultaneous $\mathbb{Q}$-smoothing of (U_i, D_i) by the description in Section 4.2. Note that $\phi_i(\eta_i) \neq 0$ by Claim 4.4. To lift $\phi_i(\eta_i)$ to $H^1(U, \Omega^2_U(\log D_U))$, we need the following claim.

Claim 4.5. $\beta_i \circ \phi_i = 0$.

Proof of Claim. $\beta_i \circ \phi_i$ is the $\mathbb{Z}_m$-invariant part of a composite of the homomorphisms

$$H^1(V_i', \Omega^2_{V_i'}(\log \Delta_i')) \to H^2_{F_i}(\tilde{V}_i, \Omega^2_{\tilde{V}_i}(\log \tilde{\Delta}_i)) \simeq H^2_{F_i}(\tilde{Y}, \Omega^2_{\tilde{Y}}(\log \tilde{\Delta})) \to H^2(\tilde{Y}, \Omega^2_{\tilde{Y}}(\log \tilde{\Delta})). \tag{23}$$

By considering its dual, it is enough to show that the $\mathbb{Z}_m$-invariant part of the homomorphism

$$\Phi_i \colon H^1(\tilde{Y}, \Omega^1_{\tilde{Y}}(\log \tilde{\Delta})(-\tilde{\Delta})) \to H^1(V_i', \Omega^1_{V_i'}(\log \Delta_i')(-\Delta_i'))$$

is zero. We show that $\Phi_i = 0$ in the following.

For a $\mathbb{Z}$-module M, we set $M_{\mathbb{C}} := M \otimes \mathbb{C}$. Let $\mathcal{K}_{(\tilde{Y},\tilde{\Delta})}$ be a sheaf of groups defined by an exact sequence

$$1 \to \mathcal{K}_{(\tilde{Y},\tilde{\Delta})} \to \mathcal{O}^*_{\tilde{Y}} \to \mathcal{O}^*_{\tilde{\Delta}} \to 1.$$

We have a commutative diagram with two horizontal exact sequences

$$\begin{tikzcd}
0 \arrow[r] & H^1(\tilde{Y}, \Omega^1_{\tilde{Y}}(\log \tilde{\Delta})(-\tilde{\Delta})) \arrow[r] & H^1(\tilde{Y}, \Omega^1_{\tilde{Y}}) \arrow[r] & H^1(\tilde{\Delta}, \Omega^1_{\tilde{\Delta}}) \\
0 \arrow[r] & H^1(\tilde{Y}, \mathcal{K}_{(\tilde{Y},\tilde{\Delta})})_{\mathbb{C}} \arrow[r] \arrow[u, "\epsilon"] & H^1(\tilde{Y}, \mathcal{O}^*_{\tilde{Y}})_{\mathbb{C}} \arrow[r] \arrow[u, "\delta_{\tilde{Y}}"] & H^1(\tilde{\Delta}, \mathcal{O}^*_{\tilde{\Delta}})_{\mathbb{C}}, \arrow[u, "\delta_{\tilde{\Delta}}"]
\end{tikzcd}$$

where the injectivity follows since we see that $H^0(\tilde{\Delta}, \Omega^1_{\tilde{\Delta}}) = 0$ and that $H^0(\tilde{Y}, \mathcal{O}^*_{\tilde{Y}}) \to H^0(\tilde{\Delta}, \mathcal{O}^*_{\tilde{\Delta}})$ is surjective. We see that $\delta_{\tilde{Y}}$ is an isomorphism and $\delta_{\tilde{\Delta}}$ is injective since we have $H^i(\tilde{Y}, \mathcal{O}_{\tilde{Y}}) = 0$ for $i = 1, 2$ and $H^1(\tilde{\Delta}, \mathcal{O}_{\tilde{\Delta}}) = 0$. Hence we see that ϵ is an isomorphism.

Set $\mathcal{K}_{(V_i',\Delta_i')} := \mathcal{K}_{(\tilde{Y},\tilde{\Delta})}|_{V_i'}$. We have a commutative diagram

$$\begin{tikzcd}
H^1(\tilde{Y}, \Omega^1_{\tilde{Y}}(\log \tilde{\Delta})(-\tilde{\Delta})) \arrow[d] & H^1(\tilde{Y}, \mathcal{K}_{(\tilde{Y},\tilde{\Delta})})_{\mathbb{C}} \arrow[l, "\simeq"'] \arrow[d, "\Phi_i'"] \\
H^1(V_i', \Omega^1_{V_i'}(\log \Delta_i')(-\Delta_i')) & H^1(V_i', \mathcal{K}_{(V_i',\Delta_i')})_{\mathbb{C}}. \arrow[l]
\end{tikzcd}$$

Hence it is enough to show that $\Phi_i' = 0$. Moreover we have a commutative diagram

$$\begin{tikzcd}
H^1(\tilde{Y}, \mathcal{K}_{(\tilde{Y},\tilde{\Delta})})_{\mathbb{C}} \arrow[r, hook] \arrow[d] & H^1(\tilde{Y}, \mathcal{O}^*_{\tilde{Y}})_{\mathbb{C}} \arrow[d] \\
H^1(V_i', \mathcal{K}_{(V_i',\Delta_i')})_{\mathbb{C}} \arrow[r, hook] & H^1(V_i', \mathcal{O}^*_{V_i'})_{\mathbb{C}}.
\end{tikzcd}$$

Since ν is an isomorphism outside $\operatorname{Sing} \Delta$, Claim 4.5 follows from the following claim.

Claim 4.6. Let $r_{\tilde{\Delta}} \colon \operatorname{Pic} \tilde{Y} \to \operatorname{Pic} \tilde{\Delta}$ be the restriction morphism as above. Then $\operatorname{Ker} r_{\tilde{\Delta}} \simeq H^1(\tilde{Y}, \mathcal{K}_{(\tilde{Y},\tilde{\Delta})})$ is generated by ν-exceptional divisors.

Proof of Claim 4.6. It is enough to show that

$$r_{\Delta} \colon \operatorname{Cl} Y \to \operatorname{Cl} \Delta$$

is injective. Indeed we have a commutative diagram

$$\begin{tikzcd} \operatorname{Cl}\tilde{Y} \arrow[r, "r_{\tilde{\Delta}}"] \arrow[d, "\nu_*"'] & \operatorname{Cl}\tilde{\Delta} \arrow[d, "(\nu_\Delta)_*"] \\ \operatorname{Cl} Y \arrow[r, "r_\Delta"] & \operatorname{Cl}\Delta \end{tikzcd}$$

and, if r_Δ is injective, can see that

$$\operatorname{Ker} r_{\tilde{\Delta}} \subset \operatorname{Ker}(\nu_\Delta)_* \circ r_{\tilde{\Delta}} = \operatorname{Ker} r_\Delta \circ \nu_* = \operatorname{Ker} \nu_*$$

and $\operatorname{Ker}\nu_*$ is generated by ν-exceptional divisors.

Let m be a sufficiently large integer such that $m\Delta$ is very ample. By [22, Theorem 1], there exists a very general smooth element $\Delta_m \in |m\Delta|$ which is disjoint with $\operatorname{Sing}\Delta$ and

$$r_{\Delta_m} \colon \operatorname{Cl} Y \to \operatorname{Cl}\Delta_m$$

is an isomorphism. Take $A \in \operatorname{Ker} r_\Delta$. Then we have $A \cdot \Delta = 0$ as a rational equivalence class of a cycle on Y. Then we have

$$A \cdot \Delta_m = 0$$

as a rational equivalence class on Y.

We show that $A|_{\Delta_m} = 0 \in \operatorname{Cl}\Delta_m$ as follows. It is enough to show that $A|_{\Delta_m}$ is numerically trivial on Δ_m since $H^1(\Delta_m, \mathcal{O}_{\Delta_m}) = 0$. Let $\Gamma \in \operatorname{Cl}\Delta_m$ be any element. Since r_{Δ_m} is an isomorphism, there exists $F \in \operatorname{Cl} Y$ such that $F|_{\Delta_m} = \Gamma$. We have

$$A|_{\Delta_m} \cdot \Gamma = (A \cdot \Delta_m) \cdot F = 0$$

by the intersection theory. Indeed $A \cdot \Delta_m$ is a sum of several curves which are regularly immersed since $\Delta_m \cap \operatorname{Sing} Y = \emptyset$. Hence $A|_{\Delta_m} = 0 \in \operatorname{Cl}\Delta_m$ and we get $A = 0 \in \operatorname{Cl} Y$ since $\operatorname{Cl} Y \xrightarrow{\simeq} \operatorname{Cl}\Delta_m$. Thus we get r_Δ is injective and we get Claim 4.6. □

Hence we see that $\Phi'_i = 0$. □

By Claim 4.5, we have $\beta_i(\phi_i(\eta_i)) = 0$. Thus there exists $\eta \in H^1(U, \Omega^2_U(\log D'))$ such that $\psi_i(\eta) = \phi_i(\eta_i)$ for each i. Then $p_{\Delta_i}(p_{U_i}(\eta) - \eta_i) \in p_{\Delta_i}(\operatorname{Ker}\tau_i) \subset \operatorname{Im}(\nu_{\Delta_i})_* = 0$ by Claim 4.4. Hence we have

$$p_{\Delta_i}(p_{U_i}(\eta)) = p_{\Delta_i}(\eta_i). \tag{24}$$

By Theorem 2.10, there exists a deformation $f\colon \mathcal{X} \to \Delta^1$ induced by η and this is a $\mathbb{Q}$-smoothing by (24) since this induces a smoothing of Δ_i, V_i and induces a deformation of U_i to a 3-fold with only quotient singularities.

□

Example 4.7. We give an example of a $\mathbb{Q}$-Fano 3-fold X such that $|-K_X|$ does not contain a Du Val elephant ([2, 4.8.3]).

Let $S_{14} \subset \mathbb{P}(2,2,3,7)$ be the surface defined by a polynomial $w^2 = y_1^3 y_2^4 - y_1 z^4$. Then S_{14} has an elliptic singularity at $[0 : 1 : 0 : 0]$. Let $X_{14} \subset \mathbb{P}(1,2,2,3,7)$ be suitable extension of S_{14} by adding several terms including x. Then we see that X_{14} is terminal and $|-K_X| = \{S_{14}\}$ with non Du-Val singularity. This (X, D) admits a simultaneous $\mathbb{Q}$-smoothing since X is a quasismooth well-formed weighted hypersurface.

4.4. **Genus bound for primary $\mathbb{Q}$-Fano 3-folds.**

Definition 4.8. Let X be a $\mathbb{Q}$-Fano 3-fold. Let $\tilde{\mathrm{Cl}} X$ be the quotient of the divisor class group $\mathrm{Cl}\, X$ by its torsion part. X is called *primary* if

$$\tilde{\mathrm{Cl}} X \simeq \mathbb{Z} \cdot [-K_X].$$

Takagi [29] proved the following theorem on the genus bound of certain primary $\mathbb{Q}$-Fano 3-folds.

Theorem 4.9. ([29, Theorem 1.5]) *Let X be a primary $\mathbb{Q}$-Fano 3-fold with only terminal quotient singularities. Assume that X is non-Gorenstein and $|-K_X|$ contains an element with only Du Val singularities.*

Then $h^0(X, -K_X) \leq 10$.

By combining his result and our results, we get the following genus bound.

Theorem 4.10. *Let X be a primary $\mathbb{Q}$-Fano 3-fold. Assume that X is non-Gorenstein and $|-K_X|$ contains an element with only Du Val singularities.*

Then $h^0(X, -K_X) \leq 10$.

Proof. By Theorem 1.9, there is a deformation $\mathcal{X} \to \Delta^1$ of X such that $\mathcal{X}_t$ has only quotient singularities and $|-K_{\mathcal{X}_t}|$ contains an element with only Du Val singularities for $t \neq 0$. By Theorem 5.28 of [12], we have $h^0(X, -K_X) = h^0(\mathcal{X}_t, -K_{\mathcal{X}_t})$. By Theorem 4.9, we have

$$h^0(X, -K_X) = h^0(\mathcal{X}_t, -K_{\mathcal{X}_t}) \leq 10.$$

□

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